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Decomposition Method for Solving Weakly Coupled Algebraic Riccati Equation

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I. Introduction

THE linear weakly coupled systems have been studied in different setups by many researchers.[1-15] The main equation of the linear optimal control theory—the Riccati equation—can be obtained from the Hamiltonian matrix. For weakly coupled systems, the Hamiltonian matrix retains the weakly coupled form by interchanging some of the state and costate variables so that it can be block diagonalized via the decoupling transformation introduced in Ref. 13. The main idea of this paper is to obtain the solution of the global algebraic Riccati equation from two decoupled reduced-order subsystems, both leading to the nonsymmetric algebraic Riccati equations that can be solved simultaneously. It has been shown that such a solution exists under stabilizability-detectability conditions imposed on both subsystems.

II. Exact Reduced-Order Algebraic Riccati Equations

Consider the linear weakly coupled system

$$\dot{x}_1 = A_1x_1 + \epsilon A_2x_2 + B_1u_1 + \epsilon B_2u_2, \quad x_1(t_0) = x_{10}$$

$$\dot{x}_2 = \epsilon A_3x_1 + A_4x_2 + \epsilon B_3u_1 + B_4u_2, \quad x_2(t_0) = x_{20} \tag{1}$$

with

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = D\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} D_1 & \epsilon D_2 \\ \epsilon D_3 & D_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2}$$

where $x_i \epsilon R^{n_i}$, $u_i \epsilon R^{m_i}$, $z_i \epsilon R^{r_i}$, $i = 1, 2$, are state, control, and output variables, respectively. The system matrices are of appropriate dimensions and, in general, they are bounded functions of a small coupling parameter ϵ.[10-12] In this paper we will assume that all given matrices are constant.

With Eqs. (1) and (2), consider the performance criterion

$$J = \frac{1}{2}\int_{t_0}^{\infty}\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T D^TD\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^T R\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right\} dt \tag{3}$$

with positive definite R, which has to be minimized. It is assumed that matrix R has the weakly coupled structure, that is,

$$R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} \tag{4}$$

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The optimal closed-loop control law has the very well-known form[16]

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -R^{-1}\begin{bmatrix} B_1 & \epsilon B_2 \\ \epsilon B_3 & B_4 \end{bmatrix}^T P\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -R^{-1}B^TPx \tag{5}$$

where P satisfies the algebraic Riccati equation given by

$$0 = PA + A^TP + Q - PSP \tag{6}$$

with

$$A = \begin{bmatrix} A_1 & \epsilon A_2 \\ \epsilon A_3 & A_4 \end{bmatrix}, \quad S = BR^{-1}B^T = \begin{bmatrix} S_1 & \epsilon S_2 \\ \epsilon S_2^T & S_3 \end{bmatrix} \tag{7}$$

and

$$Q = D^TD = \begin{bmatrix} Q_1 & \epsilon Q_2 \\ \epsilon Q_2^T & Q_3 \end{bmatrix} \tag{8}$$

The open-loop optimal control problem of Eqs. (1-4) has the solution given by

$$u(t) = -R^{-1}B^Tp(t) \tag{9}$$

where $p(t) \epsilon R^{n_1+n_2}$ is a costate variable satisfying[16]

$$\begin{bmatrix} \dot{x}(t) \\ \dot{p}(t) \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x(t) \\ p(t) \end{bmatrix} \tag{10}$$

Partitioning p into $p_1 \epsilon R^{n_1}$ and $p_2 \epsilon R^{n_2}$ such that $p = [p_1^T \; p_2^T]$, and rearranging rows in Eq. (10), we can get

$$\begin{bmatrix} \dot{x}_1 \\ \dot{p}_1 \\ \dot{x}_2 \\ \dot{p}_2 \end{bmatrix} = \begin{bmatrix} T_1 & \epsilon T_2 \\ \epsilon T_3 & T_4 \end{bmatrix}\begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} \tag{11}$$

where T_i's, $i = 1, 2, 3, 4$, are given by

$$T_1 = \begin{bmatrix} A_1 & -S_1 \\ -Q_1 & -A_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} A_2 & -S_2 \\ -Q_2 & -A_3^T \end{bmatrix}$$

$$T_3 = \begin{bmatrix} A_3 & -S_2^T \\ -Q_2^T & -A_2^T \end{bmatrix}, \quad T_4 = \begin{bmatrix} A_4 & -S_3 \\ -Q_3 & -A_4^T \end{bmatrix} \tag{12}$$

Introducing a notation

$$\begin{bmatrix} x_1 \\ p_1 \end{bmatrix} = w, \quad \begin{bmatrix} x_2 \\ p_2 \end{bmatrix} = \lambda \tag{13}$$

and applying the transformation introduced in Ref. 13

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = K^{-1}\begin{bmatrix} w \\ \lambda \end{bmatrix} \tag{14}$$

$$K = \begin{bmatrix} I & -\epsilon L \\ \epsilon H & I - \epsilon^2 HL \end{bmatrix}, \quad K^{-1} = \begin{bmatrix} I - \epsilon^2 LH & \epsilon L \\ -\epsilon H & I \end{bmatrix} \tag{15}$$

where L and H satisfy

$$T_1L + T_2 - LT_4 - \epsilon^2 LT_3L = 0 \tag{16}$$

$$H(T_1 - \epsilon^2 LT_3) - (T_4 + \epsilon^2 T_3L)H + T_3 = 0 \tag{17}$$

will produce a decoupled form

$$\dot{\eta} = (T_1 - \epsilon^2 LT_3)\eta \tag{18}$$

$$\dot{\xi} = (T_4 + \epsilon^2 T_3L)\xi \tag{19}$$

The rearrangement of states in Eq. (11) is done by using a permutation matrix E of the form

$$\begin{bmatrix} x_1 \\ p_1 \\ x_2 \\ p_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \\ p_2 \end{bmatrix} = E \begin{bmatrix} x \\ p \end{bmatrix} \tag{20}$$

Combining Eqs. (14) and (20), we obtain the relationship between the original coordinates and the new ones:

$$\begin{bmatrix} \eta_1 \\ \xi_1 \\ \eta_2 \\ \xi_2 \end{bmatrix} = E^T K^{-1} E \begin{bmatrix} x \\ p \end{bmatrix} = \Pi \begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \Pi_1 & \Pi_2 \\ \Pi_3 & \Pi_4 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \tag{21}$$

Since $p = Px$, where P satisfies the algebraic Riccati equation (6), it follows that

$$\begin{bmatrix} \eta_1 \\ \xi_1 \end{bmatrix} = (\Pi_1 + \Pi_2 P)x, \qquad \begin{bmatrix} \eta_2 \\ \xi_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)x \tag{22}$$

In the original coordinates, the required optimal solution has a closed-loop nature. We have the same attribute for the new systems in Eqs. (18) and (19), that is,

$$\begin{bmatrix} \eta_2 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \xi_1 \end{bmatrix} \tag{23}$$

Then Eqs. (22) and (23) yield

$$\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} = (\Pi_3 + \Pi_4 P)(\Pi_1 + \Pi_2 P)^{-1} \tag{24}$$

Following the same logic, we can find P reversely by introducing

$$E^T K E = \Omega = \begin{bmatrix} \Omega_1 & \Omega_2 \\ \Omega_3 & \Omega_4 \end{bmatrix} \tag{25}$$

and it yields

$$P = \left(\Omega_3 + \Omega_4 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right) \left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \right)^{-1} \tag{26}$$

The invertibility of the matrices defined in Eqs. (24) and (26) is proved in the Appendix.

Partitioning Eqs. (18) and (19) as

$$\begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} \tag{27}$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{28}$$

where

$$a_1 = A_1 + O(\epsilon^2), \quad a_2 = -S_1 + O(\epsilon^2)$$

$$a_3 = -Q_1 + O(\epsilon^2), \quad a_4 = -A^T + O(\epsilon^2) \tag{29}$$

$$b_1 = A_4 + O(\epsilon^2), \quad b_2 = -S_3 + O(\epsilon^2)$$

$$b_3 = -Q_3 + O(\epsilon^2), \quad b_4 = -A^T + O(\epsilon^2) \tag{30}$$

and using Eq. (23) yield two reduced-order nonsymmetric algebraic Riccati equations

$$0 = P_1 a_1 - a_4 P_1 - a_3 + P_1 a_2 P_1 \tag{31}$$

$$0 = P_2 b_1 - b_4 P_2 - b_3 + P_2 b_2 P_2 \tag{32}$$

From Eqs. (29) and (30), it follows that the $O(\epsilon^2)$ perturbations of the nonsymmetric algebraic Riccati equations Eqs. (31) and (32) are symmetric, namely,

$$P_1 A_1 + A_1^T P_1 + D_1^T D_1 - P_1 B_1 R_1^{-1} B_1^T P_1 + O(\epsilon^2) = 0 \tag{33}$$

$$P_2 A_4 + A_4^T P_2 + D_4^T D_4 - P_2 B_4 R_2^{-1} B_4^T P_2 + O(\epsilon^2) = 0 \tag{34}$$

Using these facts and the implicit function theorem,[17] the existence of the unique solutions of Eqs. (31) and (32) is guaranteed under the following lemma.

Lemma: Existence of Solutions to Eqs. (31) and (32)

If both (A_1, B_1, D_1) and (A_4, B_4, D_4) are stabilizable detectable, then it exists $\epsilon_0 > 0$ such that for every $\epsilon \le \epsilon_0$ the solutions to Eqs. (31) and (32) exist.

Two numerical methods can be proposed for solving Eqs. (31) and (32), namely, the fixed-point and Newton method similar to those developed in Refs. 12 and 13. The Newton method leads to the following recursive scheme:

$$P_1^{(i+1)} \left[a_1 + a_2 P_1^{(i)} \right] - \left[a_4 - P_1^{(i)} a_2 \right] P_1^{(i+1)}$$
$$= a_3 + P_1^{(i)} a_2 P_1^{(i)} \tag{35}$$

where the initial condition is obtained from

$$P_1^{(0)} A_1 + A_1^T P_1^{(0)} + Q_1 - P_1^{(0)} S_1 P_1^{(0)} = 0 \tag{36}$$

Similar formulas hold for Eq. (32).

It is interesting to point out that the proposed method is not applicable for the differential Riccati equation of weakly coupled systems because there is no way to find the terminal conditions for the reduced-order nonsymmetric differential Riccati equations.

III. Numerical Example

To demonstrate the presented method, we have solved a fourth-order example, a satellite control problem considered in Ref. 18. Problem matrices are given by[18]

$$A = \begin{bmatrix} 0 & 0.667 & 0 & 0 \\ -0.667 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.53 \\ 0 & 0 & -1.53 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0.2 \\ 1 & 0 \\ 0.4 & 0 \\ 0 & 1 \end{bmatrix}$$

Penalty matrices Q and R are chosen as identities.

Results obtained from Eqs. (31) and (32) are given by

$$P_1 = \begin{bmatrix} 2.2201 & 0.45889 \\ 0.4410 & 1.2749 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1.5056 & 0.1947 \\ 0.22817 & 1.2782 \end{bmatrix}$$

which by the use of the formula of Eq. (26) produce

$$P = \begin{bmatrix} 2.2437 & 0.46218 & 0.13613 & -0.10735 \\ 0.46218 & 1.3456 & -0.2091 & -0.24753 \\ 0.13613 & -0.2091 & 1.5375 & 0.24817 \\ -0.10735 & -0.24753 & 0.24817 & 1.3396 \end{bmatrix}$$

Exactly the same result has been obtained by using the classical global method for solving the algebraic Riccati equation.

IV. Conclusion

The optimal steady-state, closed-loop control problem of weakly coupled systems is solved by way of the reduced-order nonsymmetric algebraic Riccati equations. Since these two Riccati equations are completely independent, they can be solved simultaneously and thus reduce the processing time for the optimal control and filtering problems of weakly coupled linear systems.

Appendix: Invertibility Proofs

According to Eqs. (21) and (25), it can be seen that

$$\Pi = \begin{bmatrix} I_{n_1} + O(\epsilon^2) & O(\epsilon) & O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & I_{n_2} & O(\epsilon) & 0 \\ O(\epsilon^2) & O(\epsilon) & I_{n_1} + O(\epsilon^2) & O(\epsilon) \\ O(\epsilon) & 0 & O(\epsilon) & I_{n_2} \end{bmatrix} \tag{A1}$$

$$\Omega = \begin{bmatrix} I_{n_1} & O(\epsilon) & 0 & O(\epsilon) \\ O(\epsilon) & I_{n_2} + O(\epsilon^2) & O(\epsilon) & O(\epsilon^2) \\ 0 & O(\epsilon) & I_{n_1} & O(\epsilon) \\ O(\epsilon) & O(\epsilon^2) & O(\epsilon) & I_{n_2} + O(\epsilon^2) \end{bmatrix} \tag{A2}$$

Therefore,

$$(\Pi_1 + \Pi_2 P) = I_{n_1+n_2} + O(\epsilon) \tag{A3}$$

$$\left(\Omega_1 + \Omega_2 \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}\right) = I_{n_1+n_2} + O(\epsilon) \tag{A4}$$

There exists $\epsilon_1 > 0$ such that for every $\epsilon \leq \epsilon_1$ the required matrices are invertible.

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Small Gain Versus Positive Real Modeling of Real Parameter Uncertainty

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Introduction

THE small gain theorem is one of the principal tools for modeling plant uncertainty. A standard representation[1] of an uncertain plant under feedback is shown in Fig. 1. The plant P and the compensator G are assumed to be known, while the plant uncertainty Δ has been "pulled out" into a fictitious feedback loop. In general, Δ may have a block-diagonal structure composed of scalar and/or matrix blocks. In this Note, we assume Δ is composed of a single matrix block. Figure 2 shows an equivalent representation with $\tilde{P}$ denoting the nominal closed-loop system. Note that r represents a fictitious input that is used as a means of representing the nominal closed loop in feedback with the uncertainty. From the small gain theorem[2] it follows that if $\tilde{P}$ and Δ are stable bounded-gain transfer functions such that $\|\Delta\|_\infty \|\tilde{P}\|_\infty < 1$ for all uncertainties Δ, then the closed-loop system is robustly stable.

Now suppose that the previous uncertainty model is used to represent constant real parameter uncertainty. The inherent conservatism of such a model can be demonstrated in two different ways. From a time-domain point of view, it is shown in Theorem 2.7 of Ref. 3 that the existence of an H_∞ norm bound is equivalent to the existence of a quadratic Lyapunov function that guarantees robust stability with respect to *time-varying* parameter variations. It is well known from the classical analysis of Hill's equation (e.g., the Mathieu equation) that time-varying parameter variations can destabilize a system even when the parameter variations are confined to a region in which constant variations are nondestabilizing.

From a frequency-domain point of view, the uncertainty block Δ satisfying an H_∞ norm bound can represent an arbitrary linear time-invariant transfer function possessing arbitrary frequency-dependent phase characteristics. A constant real parameter variation, however, at least in the scalar case, can be viewed as a transfer function that possesses a constant phase of 0 deg (if positive) or 180 deg (if negative). Thus, H_∞ modeling of real parameter uncertainty permits much larger

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